Chapter 8

Calculus

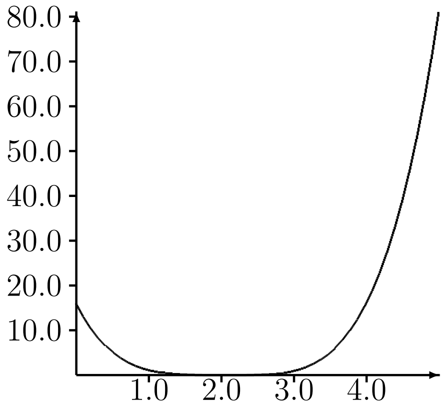
I'm very good at integral and differential calculus, I know the scientific names of beings animalculous; In short, in matters vegetable, animal, and mineral, I am the very model of a modern Major-General.
The Pirates of Penzance. Act 1.

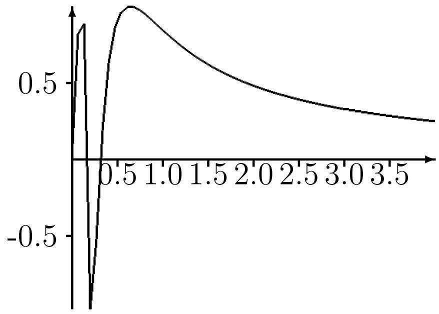
We have looked at limits of sequences, now I want to look at limits of functions. Suppose we have a function $f(x)$ defined on an interval $a \leq x \leq b$. I have a sequence $x_1, x_2, \cdots, x_n$ which tends to a limit x_0. Can I say that the sequence $f(x_1), f(x_2, \ldots, f(x_n)$ tends to ℓ and what do I mean? We normally define the limit as follows:

> We say that $f(x) \to f(x_0)$ as $x \to x_0$ if for any $\epsilon > 0$ there is a value $\delta > 0$ such that $\mid x - x_0 \mid < \delta \Rightarrow \mid f(x) - \ell \mid < \epsilon$

This is in the same spirit as our previous definition for sequences. We can be as close as we wish to the limiting value ℓ.

For example $(x-2)^4 \to 0$ as $x \to 2$. If you given me an $0 < \epsilon < 1$ then if $\mid x - 2 \mid \leq \delta$ we know $\mid (x-2)^4 - 0 \mid \leq \delta^4$. So provided $\delta \leq \epsilon$ we have a limit as $x \to 0$!

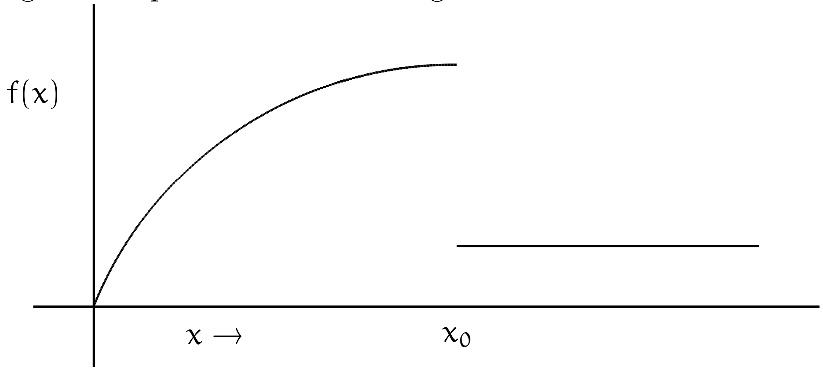




In the second case we plot $\sin(1/x)$. This starts to oscillate faster and faster as it approaches zero and (it is not quite simple to show) does not have a limit.

8.0.2 Continuity and Differentiability

We did not specify which direction we used to approach the limiting value, from above or from below. This might be important as in the diagram below where the

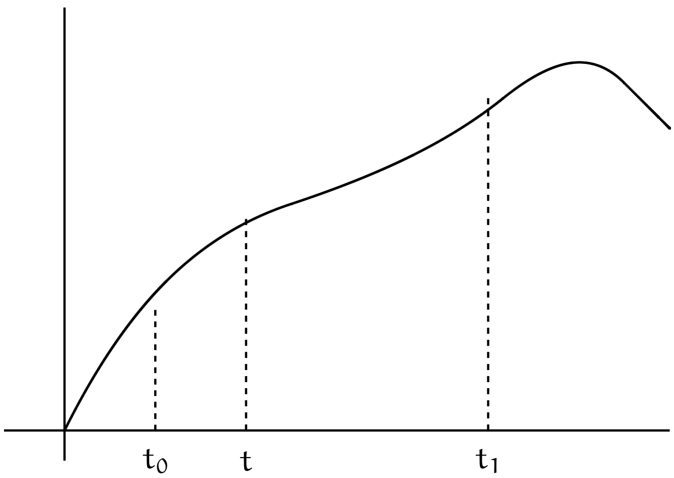


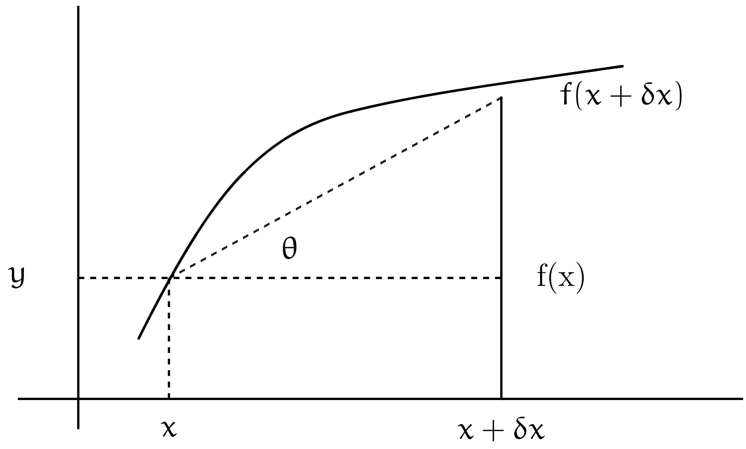
function has a *jump* at x_0.

We like *continuous* functions, these are functions where $f(x) \to f(x_0)$ as $x \to x_0$ from above and below. You can think of these as functions you can draw without lifting your pencil off the page. Continuous functions have lots of nice properties.

If we have a continuous function we might reasonably look at the *slope* of the curve at any point. This may have a real physical meaning. So suppose we have the track of a car. We might plot the distance it travels, East say, against time.

If the difference between the distance at times t_0 and t_1 is D then $D/(t_1 - t_0)$ gives the approximate speed. This is just the procedure followed by average speed cameras on roads! However what we have observed is an average speed. If we want an estimate of speed at a particular time t we need t_0 and t_1 to approach t.





If we take the times to be t and $t+\delta t$, where δt means a small extra bit of t, then we want

$$\frac{f(t+\delta t)-f(t)}{(t+\delta t-t)}$$

as δt becomes small or more explicitly

$$\frac{f(t+\delta t)-f(t)}{\delta t} \text{ as } t \to 0$$

This limit gives the *derivative* which is the slope of the curve $f(t)$ at the point t and is written $f'(t)$ or

$$\frac{df}{dx} = \lim_{\delta t \to 0} \frac{f(t+\delta t)-f(t)}{\delta t} \tag{8.1}$$

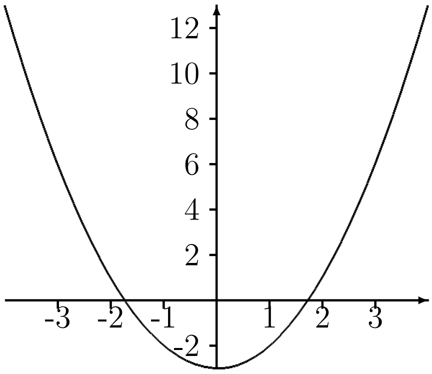
Suppose we take $y = f(t) = 3-4t$, a line with constant negative slope. Using the equation 8.1 we have

$$\frac{df}{dx} = \lim_{\delta t \to 0} \frac{3-4(t+\delta t)-3+4t}{\delta t} = \frac{-4\delta t}{\delta t} = -4$$

If we now have $y = x^2 - 3$ we have, writing x for t

$$\frac{df}{dx} = \lim_{\delta t \to 0} \frac{(x+\delta x)^2-3-x^2+3}{\delta x} = \frac{x^2+2x\delta x+(\delta x)^2-3-x^2+3}{\delta x} = \frac{2x\delta x+(\delta x)^2}{\delta x} = 2x{+}\delta x =$$

So at x=2 the slope is zero while when x is negative the slope is down and then is upwards when x is greater that zero. You might find it useful to consider the plot. Note that if we take a point on a curve and draw a straight line whose slope is $f'(x)$ this line is known as the *tangent* at x.



Of course life is too short for working out the derivatives dy/dx like this from first principles so we tend to use rules (derived from first principles).

1. $\dfrac{d}{dx}[af(x)] = a\dfrac{df}{dx}$ where a is a constant.
2. $\dfrac{d}{dx}[f(x) + g(x)] = \dfrac{df}{dx}\dfrac{dg}{dx}$
3. $\dfrac{d}{dx}[f(x)g(x)] = f(x)\dfrac{dg}{dx} + g(x)\dfrac{df}{dx}$
4. $\dfrac{d}{dx}\dfrac{1}{f(x)} = -\dfrac{1}{f^2(x)}\dfrac{df}{dx}$
5. $\dfrac{d}{dx}[x^n] = nx^{n-1}$ when $n \neq 0$ and zero otherwise.
6. $\dfrac{df(g(x))}{dx} = f^{'}(g(x))g^{'}(x)$ using $'$ for the derivative.

This set of rules makes like very easy, so

$$\frac{d}{dx}(3x^2 - 11x + 59) = 3 \times 2x - 11$$

$$\frac{d}{dx}\frac{1}{(3x^2 - 11x + 59)} = -\frac{6x - 11}{(3x^2 - 11x + 59)^2}$$

$$\frac{d}{dx}(3x^2 - 11x + 59)(x - 1) = (6x - 11)(x - 1) + (3x^2 - 11x + 59)(1)$$

Example

Suppose we would like to show that $\sin x \leq x$ for $0 \leq x \leq \pi/2$. We know that when $x = 0 \quad x = \sin x = 0$. But

$$\frac{dx}{dx} = 1 \text{ and } \frac{d \sin x}{dx} = \cos x$$

Since $\cos x \leq 1$ in the interval it implies that $\sin x$ grows more slowly than x and the result follows.

Once we move away from polynomials life gets a little more complex. In reality you need to know the derivative to be able to proceed so you need a list such as in table 8.1. Note that the derivative of $\exp(x)$ is just $\exp(x)$. So for example

Table 8.1: Table of derivatives: all logs are base e and a is a constant

Function	Derivative
$\exp(ax)$	$a\exp(ax)$
a^x	$a^x \log(a)$
$\log(ax)$	$\frac{1}{x}$
x^x	$x^x(1 + \log x)$
$\sin(ax)$	$a\cos(x)$
$\cos(ax)$	$-a\sin(x)$
$\tan(ax)$	$\frac{a}{\cos^2(x)}$

- If $y = \exp(-x^2)$ then $\dfrac{d\exp(-x^2)}{dx} = \exp(-x^2)(-2x)$
- If $y = \log(3x^2 - 4x + 1)$ then $\dfrac{d\log(3x^2 - 4x + 1)}{dx} = \dfrac{6x - 4}{(3x^2 - 4x + 1)}$

It is important to remember that the formulas only work for logarithms to base e and trigonometric functions, sin, cos etc expressed in *radians*.

higher derivatives

Since $\dfrac{dy}{dx}$ is a function we might wish to differentiate it again to get $\dfrac{d\left[\frac{dy}{dx}\right]}{dx}$ called the second derivative and written $\dfrac{d^2y}{dx^2}$. If we differentiate 4 times we write $\dfrac{d^4y}{dx^4}$ and in general

$$\frac{d^ny}{dx^n} \qquad n = 2, 3, 4, \ldots$$

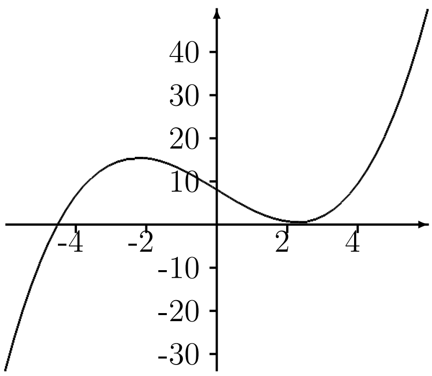
So if $y = \log(x)$ we have

$$\frac{dy}{dx} = \frac{1}{x} \quad \frac{d^2y}{dx^2} = -\frac{1}{x^2} \quad \frac{d^3y}{dx^3} = \frac{2}{x^3} \quad \frac{d^4y}{dx^4} = -\frac{6}{x^4} \ldots$$

Maxima and minima

One common use for the derivative is to find the maximum or minimum of a function. It is easy to see that if we have a maximum or minimum of a function then the derivative is zero. Consider $y = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 6x + 8$

$f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 6x + 8$



We compute $\dfrac{df}{dx} = x^2 + x - 6$ which is zero when $x^2 + x - 6 = (x+3)(x-2) = 0$ or $x = -3$ and $x = 2$ and from the plot it we see that we have found the turning points of the function. These are the local maxima and minima.

However when we step back and look at the whole picture it is possible to we have a stationary point i.e. $\dfrac{df}{dx} = 0$ which is not a turning point and hence we need a local max or minimum rule:

$\frac{dy}{dx} = 0$	
$\frac{dy}{dx} < 0$ for $x < x_0$ $\frac{dy}{dx} > 0$ for $x > x_0$	$\frac{dy}{dx} > 0$ for $x < x_0$ $\frac{dy}{dx} < 0$ for $x > x_0$
x_0 is a minimum	x_0 is a maximum
$\frac{d^2y}{dx^2} > 0$	$\frac{d^2y}{dx^2} < 0$

1. The function $f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 6x + 8$ has derivative $\frac{dy}{dx} = x^2 + x - 6$ so at $x = 2$ we have $\frac{dy}{dx} = 0$. When $x < 2$ the derivative is negative while when $x > 2$ it is positive so we have a minimum.

2. Or perhaps simpler $\frac{d^2y}{dx^2} = 2x + 1 > 0$ at $x = 2$ so we have a minimum.

3. When $x = -3$ again $\frac{dy}{dx} = 0$. For $x < -3 \frac{dy}{dx} > 0$ while when $x > -3 \frac{dy}{dx} < 0$ implying a maximum.

4. Again for simplicity $\frac{d^2y}{dx^2} = 2x + 1 < 0$ at $x = -3$ hence we have a maximum.

Example

Suppose we make steel cans. If the form of the can is a cylinder of height h and radius r the volume of the can is $V = \pi r^2 h$ and the area of the steel used is $A = 2\pi rh + 2\pi r^2$.

We want the volume to be 64cc. and hence $V = \pi r^2 h = 64$ which gives $h = 64/(\pi r^2)$. The area is therefore $A = 2\pi rh + 2\pi r^2 = 128/r^2 + 2\pi r^2$

To minimize the area we compute

$$\frac{dA}{dr} = -128/r^2 + 4\pi r$$

which is zero when $4\pi r^3 = 128$ giving $r \simeq 2.17$ and $h = 64/(\pi r^2) \simeq 4.34$.

To check that this is a minimum

$$\frac{d^2A}{dr^2} = 256/r^3 + 4\pi$$

which is positive when r is positive so we have a minimum.

The Taylor Expansion

We leave you with one useful approximation. If we have a function $f(x)$ then we have

$$f(x+a) = f(x) + a\frac{df}{dx} + \frac{a^2}{2!}\frac{df^2}{dx^2} + \ldots + \frac{a^n}{n!}\frac{df^n}{dx^n} + \ldots$$

When a is small and we evaluate the derivatives at x. For example if we take $\sin x$ the derivatives are $\cos x$, $-sinx$, $-cosx$, $sinx$, ... So at $x = 0$ since $\sin 0 = 0$ and $\cos 0 = 1$

$$\sin(a) = a - \frac{a^3}{3!} + \frac{a^5}{5!} - \frac{a^7}{7!} - \ldots$$

8.0.3 Newton-Raphson method

We now examine a method, known as the Newton-Raphson method, that makes use of the derivative of the function to find a zero of that function. Suppose we have reason to believe that there is a zero of $f(x)$ near the pointx_0. The Taylor expansion for $f(x)$ about x_0 can be written as:

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{1}{2!}(x - x_0^2 f''(x_0) + \ldots$$

If we drop the terms of this expansion beyond the first order term we have

$$f(x) = f(x_0) + (x - x_0)f'(x_0)$$

Now set $f(x) = 0$ to find the next approximation, x_1, to the zero of $f(x)$, we find:

$$f(x_1) = f(x_0) + (x_1 - x_0)f'(x_0) = 0$$

or

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

This provides us with an *iteration scheme* which may well converge on the zero of $f(x)$, under appropriate conditions.

example

Suppose we want the cube root of 2 or the value of x for which $f(x) = x^3 - 2 = 0$. Here $f'(x) = 3x^2$ so

$$x_1 = x_0 - \frac{x_0^3 - 2}{3x_0^2}$$

Starting with $x_0 = 1$ we have $x_1 = 1.333333$ and using this value for x_0 we get $x_1 = 1.263889$. The steps are laid out below

Step	Estimate
0	1
1	1.333333
2	1.263889
3	1.259933
4	1.259921

Or suppose $f(x) = \sin x - \cos x$ then $f'(x) = \cos x + \sin x$ and so

$$x_1 = x_0 - \frac{\sin x_0 - \cos x_0}{\cos x_0 + \sin x_0}$$

then starting with $x_0 = 1$ we have

Step	Estimate
1	1
2	0.7820419
3	0.7853982
4	0.7853982
5	0.7853982

To examine the conditions under which this iteration converges, we consider the iteration function

$$g(x) = x - \frac{f(x)}{f'(x)}$$

whose derivative is:

$$g^{'}(x) = 1 - \frac{(f^{'}(x))^2 - f(x)f^{''}(x)}{(f^{'}(x))^2} = \frac{f(x)f^{''}(x)}{(f^{'}(x))^2}$$

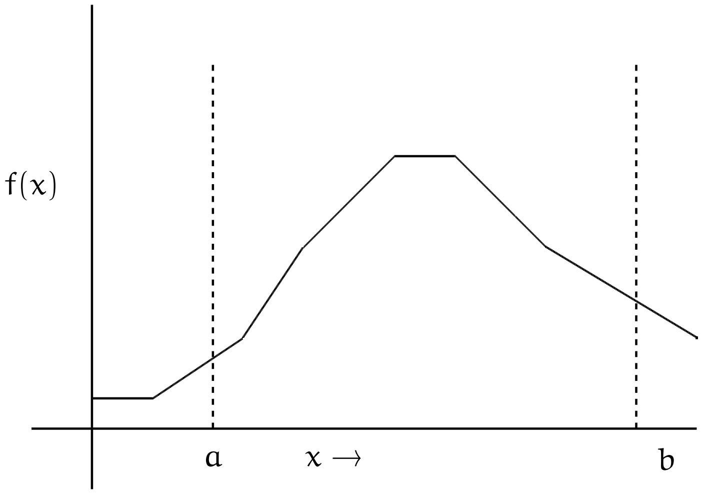
At the actual zero, $f(x) = 0$, so that as long as $f^{'}(x) = 0$, we have $g^{'}(x) = 0$ at the zero of $f(x)$. In addition we would like the iteration function to get smaller, that is $\mid g^{'}(x) \mid < 1$. We conclude that the Newton-Raphson method converges in the interval where.

$$\frac{f(x)f^{''}(x)}{(f^{'}(x))^2} < 1$$

Step	Estimate
0	1
1	1.333333
2	1.263889
3	1.259933
4	1.259921

8.0.4 Integrals and Integration

Many important problems can be reduced to finding the area under a curve between two points a and b



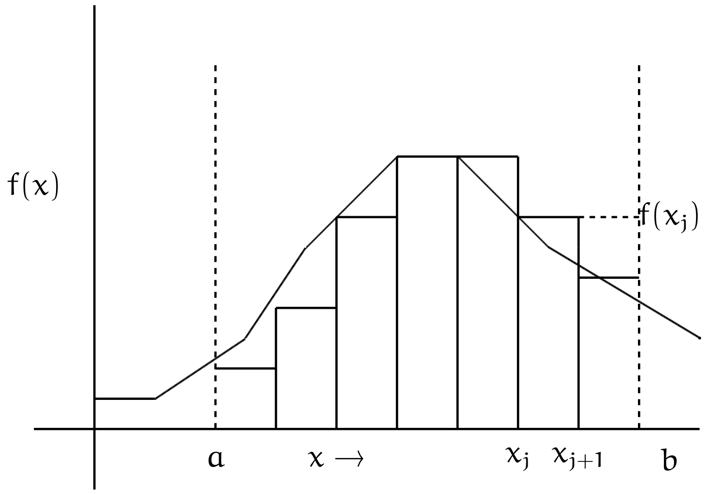
The obvious idea is to split the area into small rectangles and sum the area of these. So if we take the rectangle between x_j and x_{j+1} this has a height of $f(x_j)$ and an area of $f(x_j)(x_{j+1} - x_j)$. If we add all such rectangles this gives an gives an approximation to the area. We do better when the width of the rectangles gets small so if we choose all the widths as δ our approximation is

$$\sum f(x_j)\delta x \text{ for } a = x_1, x_2, \ldots, x_n = b$$

When we shrink δx to zero we have the area we need and write

$$\int_a^b f(x)dx$$

The $\int$ sign was originally a capital S, for sum.



We avoid technicalities and define the *definite integral* of a function$f(x)$ between a and b as

$$\int_a^b f(x)dx$$

which is the area under the curve, see figure 8.1 Using the idea of areas we have

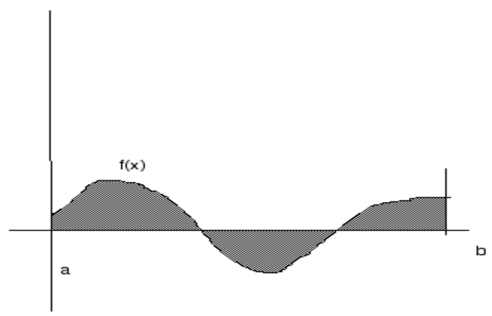


Figure 8.1: Areas under $f(x)$

some rule for integrals

1. If $a \leq c \leq b$ then $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$
2. For a constant c $\int_a^b cf(x)dx = c\int_a^b f(x)dx$
3. For two functions $f(x)$ and $g(x)$ $\int_a^b c\,(f(x) + g(x))\,dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

Perhaps the most important result about integration is the fundamental theorem of calculus. It is easy to follow, if not to prove. Suppose we have a function $f(x)$ and we define

$$F(x) = \int_a^x f(t)dt$$

Then

$$\frac{dF(x)}{dx} = \frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x)$$

In other words integration is rather like the reverse of differentiation. We need to be a bit careful so define $F(x)$ as the *primitive* of $f(x)$ if

$$\frac{dF(x)}{dx} = f(x)$$

So $\log x$ is a primitive for $1/x$ as is $\log x + 23$. The primitive is normally called the indefinite integral $\int f(x)dx$ of $f(x)$ and is defined up to a constant, so $\int f(x)dx = F(x) + constant$

If the limits of the integration exist, say a and b then we have the definite integral

$$\int_a^b f(x)dx = F(b) - F(a) \tag{8.2}$$

We can of course spend time looking at functions which differentiate to what we want. Normally however we use tables (or our memory) So

$f(x)$	$F(x) = \int f(x)dx$
x^n $(n \neq -1)$	$x^n/(n+1)$
$1/x$	$\log x$
$\exp(ax)$	$\exp(ax)/a$
$\log x$	$x \log x - x$
a^x	$a^x/\log a$
$\sin(ax)$	$-\cos(ax)/a$
$\cos(ax)$	$\sin(ax)/a$
$1/\sqrt{a^2 - x^2}$	$\sin^{-1}(x/a)$ $(-1 < x < a)$
$1/(a^2 - x^2)$	$\tan^{-1}(x/a)$

Example

1. $\int x^2 dx = x^3/3 + constant$
2. $\int_{-2}^{3} x^2 dx = \left[x^3/3\right]_{-2}^{3} = (3)^3/3 - (-2)^3/3 = (27+8)/3$
3. $\int_1^{10} = 1/x dx = [\log x]_1^{10} = \log 10 - \log 1 = 2.30\ldots - 0$
4. $\int_0^{1/2} dx/\sqrt{1-x^2} = [\sin_1(x)]_1^{10} = \sin^{-1}(1/2) - \sin_{-1}(0) = \pi/6$

Exercises

Evaluate the following integrals and check your solutions by differentiating.

1. $\int x^3 dx$
2. $\int 1/x^2 dx$
3. $\int (25 + x^2)^{-1} dx$

Evaluate

1. $\int_3^7 \log x \, dx$
2. $\int_1^2 x^{-3/2} dx$
3. $\int_a^{2a} (a^2 + x^2)^{-1} dx$